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Remarks on lower bounds for the maximal existence time to the 3D rotating Euler equations

By

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Abstract

We consider the long time existence of classical solutions to the 3D incompressible Euler equations with the Coriolis force. We shall give lower bounds for the maximal existence time of classical solutions in terms of the speed of rotation.

§ 1. Introduction

Let us consider the initial value problem of the 3D rotating Euler equations, describing the motion of perfect incompressible fluids in the rotational framework.

$$(\mathrm{E}_\Omega) \qquad \begin{cases} \partial_t u + \Omega e_3 \times u + (u \cdot \nabla) u + \nabla p = 0 & t > 0,\ x \in \mathbb{R}^3, \\ \operatorname{div} u = 0 & t > 0,\ x \in \mathbb{R}^3, \\ u(0,x) = \phi(x) & x \in \mathbb{R}^3. \end{cases}$$

Here, $u = (u_1(t,x), u_2(t,x), u_3(t,x))$ denotes the unknown velocity field and $p = p(t,x)$ denotes the unknown scalar pressure, while $\phi = (\phi_1(x), \phi_2(x), \phi_3(x))$ denotes the given initial velocity fields satisfying $\operatorname{div} \phi = 0$. The constant $\Omega \in \mathbb{R}$ corresponds to the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$, which is called the Coriolis parameter.

In this paper, we shall consider the long time existence of the unique classical solution to (E_Ω) for the initial velocity belonging to $H^s(\mathbb{R}^3)$ with $s \geqslant 7/2$. More precisely, we show that for given initial velocity $\phi \in H^s(\mathbb{R}^3)$ with $s \geqslant 7/2$ satisfying

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$\operatorname{div}\phi = 0$ and given time $0 < T < \infty$, there exists a positive number $\Omega_{\phi,T}$ such that the 3D rotating Euler equation (E_Ω) possesses a unique classical solution u on the given time interval $[0,T]$ provided $|\Omega| \geqslant \Omega_{\phi,T}$. In particular, we shall characterize an upper bound of the minimal speed of rotation $\Omega_{\phi,T}$ which ensures the long time existence to (E_Ω) by means of the norm of initial velocity $\|\phi\|_{H^s}$ and the time T. As a by-product of our characterization, we also obtain a lower bound for the maximal existence time of the classical solutions in terms of the speed of rotation $|\Omega|$.

Let $\mathbb{P} := (\delta_{jk} + R_j R_k)_{1\leqslant j,k\leqslant 3}$ be the Helmholtz projection onto the divergence-free vector fields, where R_j denotes the Riesz transform in $\mathbb{R}^3$. Applying the projection $\mathbb{P}$ to both sides of the first equation of (E_Ω), we obtain the following evolution equations for the velocity fields:

$$(\mathrm{E}'_\Omega) \qquad \begin{cases} \partial_t u + \Omega\mathbb{P}\,(e_3 \times u) + \mathbb{P}(u\cdot\nabla)u = 0 & t>0,\ x\in\mathbb{R}^3, \\ \operatorname{div} u = 0 & t>0,\ x\in\mathbb{R}^3, \\ u(0,x) = \phi(x) & x\in\mathbb{R}^3. \end{cases}$$

We now review the local existence results on the original Euler equations for $\Omega = 0$. Kato [14] proved that for given integer $m \in \mathbb{Z}$ with $m > 5/2$ and for given initial velocity $\phi \in H^m(\mathbb{R}^3)$ satisfying the compatibility condition $\operatorname{div}\phi = 0$, there exists a positive time $T = T(\|\phi\|_{H^m})$ such that (E'_0) admits a unique classical solution u in the class $C([0,T];H^m(\mathbb{R}^3)) \cap C^1([0,T];H^{m-1}(\mathbb{R}^3))$. Kato-Ponce [15] extended this result to the Sobolev spaces $W^{s,p}(\mathbb{R}^3)$ of the fractional order for $s > 3/p+1, 1<p<\infty$. Chae [5] and Chen-Miao-Zhang [8] gave further extensions to the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^3)$ with $s > 3/p+1, 1 < p,q < \infty$. Chae [6] also obtained the local well-posedness in the Besov spaces $B^s_{p,q}(\mathbb{R}^3)$ with $s > 3/p+1, 1<p<\infty, 1\leqslant q\leqslant\infty$ or $s = 3/p+1, 1<p<\infty, q=1$. Pak-Park [19] extended these results to the Besov space $B^1_{\infty,1}(\mathbb{R}^3)$.

For the large Coriolis parameter $|\Omega|$, Dutrifoy [11] showed the long time existence of classical solutions for the initial data in $H^s(\mathbb{R}^3)$ with $s > 7/2$ or $B^{7/2}_{2,1}(\mathbb{R}^3)$, and proved the asymptotics of solutions to vortex patches or Yudovich solutions as the Rossby number goes to zero for some particular initial data. Similar results are obtained for the quasigeostrophic systems by Dutrifoy [10] and Charve [7]. Also, they gave a lower bound for the maximal existence time T_Ω of the solution to (E'_Ω) by the double logarithmic order as $T_\Omega \gtrsim \log\log|\Omega|$. Koh-Lee-Takada [16] obtained the optimal range of the Strichartz estimate for the linear propagator associated with the Coriolis force, and showed the long time existence results for the initial data in $H^s(\mathbb{R}^3)$ $(s > 7/2)$ together with the single logarithmic lower bound $T_\Omega \gtrsim \log|\Omega|$ for the maximal existence time. For the periodic setting in $\mathbb{T}^3$, we refer to Babin-Mahalov-Nicolaenko [1] [2] [3].

In this paper, we shall establish an improved relation between the given data

$(\phi, T) \in H^s(\mathbb{R}^3) \times (0, \infty)$ with $s > 7/2$ and the minimal speed of rotation $\Omega_{\phi,T}$ for the long time existence of solutions to (E'_Ω), which also gives a lower bound for the maximal existence time T_Ω by a polynomial growth of the Coriolis parameter.

Our first result reads as follows:

Theorem 1.1. *Let $s \in \mathbb{R}$ satisfy $s > 5/2$. Then, for every $\phi \in H^{s+1}(\mathbb{R}^3)$ satisfying $\operatorname{div} \phi = 0$ and for every $0 < T < \infty$, there exists a positive constant $\Omega_{\phi,T}$ depending on s, T and $\|\phi\|_{H^{s+1}}$ such that if $|\Omega| \geqslant \Omega_{\phi,T}$ then (E'_Ω) possesses a unique classical solution u in the class*

$$u \in C([0,T]; H^{s+1}(\mathbb{R}^3)) \cap C^1([0,T]; H^s(\mathbb{R}^3)). \tag{1.1}$$

In particular, let $\Omega^*_{\phi,T}$ be the infimum of the set of $|\Omega| \geqslant 0$ such that (E'_Ω) admits a unique classical solution u in the class (1.1). Then, for $2 < q < \infty$ there exist positive constants $C_1 = C_1(s,q)$ and $C_2 = C_2(s)$ such that

$$\Omega^*_{\phi,T} \leqslant C_1 T^{q-1} \left\{ \|\phi\|_{H^{s+1}} \left(1 + e^{C_2 T \|\phi\|_{H^{s+1}}}\right) \right\}^q . \tag{1.2}$$

Remark. It follows from the characterization (1.2) in Theorem 1.1 that the maximal existence time $T_\Omega \geqslant 1$ of the solution to (E'_Ω) has a lower bound

$$T_\Omega \geqslant \frac{C_{s,q} |\Omega|^{\frac{1}{2q}}}{\|\phi\|^{\frac{1}{2}}_{H^{s+1}} (1 + \|\phi\|_{H^{s+1}})^{\frac{1}{2}}}$$

for sufficiently large Coriolis parameter $|\Omega|$ with some positive constant $C_{s,q}$ depending on s and q. This improvement is due to the use of a single exponential estimate for the blow-up criterion (see (4.1) in Lemma 4.2 below).

Next, we consider the case $\phi \in H^{\frac{7}{2}}(\mathbb{R}^3)$. We remark that the following Theorem 1.2 has already been announced in our previous work [20].

Theorem 1.2 ([20]). *For every $\phi \in H^{\frac{7}{2}}(\mathbb{R}^3)$ satisfying $\operatorname{div} \phi = 0$ and for every $0 < T < \infty$, there exists a positive constant $\Omega_{\phi,T}$ depending on T and $\|\phi\|_{H^{\frac{7}{2}}}$ such that if $|\Omega| \geqslant \Omega_{\phi,T}$ then (E'_Ω) possesses a unique classical solution u in the class*

$$u \in C([0,T]; H^{\frac{7}{2}}(\mathbb{R}^3)) \cap C^1([0,T]; H^{\frac{5}{2}}(\mathbb{R}^3)). \tag{1.3}$$

In particular, let $\Omega^*_{\phi,T}$ be the infimum of the set of $|\Omega| \geqslant 0$ such that (E'_Ω) admits a unique classical solution u in the class (1.3). Then, for $2 < q < \infty$ there exist a positive absolute constant $C_* = C_*(7/2)$ and a positive constant C_q depending on q such that

$$\Omega^*_{\phi,T} \leqslant C_q T^{q-1} \left\{ \|\phi\|_{H^{\frac{5}{2}}} + T(\|\phi\|_{H^{\frac{7}{2}}} + e)^{C_*} e^{C_* T} \right\}^q . \tag{1.4}$$

Remark. It follows from the characterization (1.4) in Theorem 1.2 that the maximal existence time $T_\Omega \geqslant 1$ has a lower bound

$$T_\Omega \geqslant \frac{C'_q}{\log(\|\phi\|_{H^{\frac{7}{2}}} + e)} \log\left(\frac{|\Omega|}{C''_q}\right)$$

for sufficiently high speed of rotation $|\Omega|$ with some positive constants C'_q and C''_q depending on q. This single logarithmic order is due to the use of the logarithmic Sobolev inequality for the blow-up criterion (see Lemma 2.2 and (4.2) in Lemma 4.2 below).

This paper is organized as follows. In Section 2, we recall the definitions of function spaces and some basic inequalities in these spaces. In Section 3, we derive the solution formula for the linear vorticity equations and recall the space-time estimates for the linear propagator. In Section 4, we state the uniform local existence result and the blow-up criteria. In Sections 5 and 6, we present the proofs of Theorems 1.1 and 1.2, respectively.

Throughout this paper, we denote by C the constants which may differ from line to line. In particular, $C = C(\cdot, \dots, \cdot)$ will denote the constant which depends only on the quantities appearing in parentheses. For $A, B \geqslant 0$, $A \lesssim B$ means that there exists some positive constant C such that $A \leqslant CB$. Also, $A \gtrsim B$ is defined in the same way. $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$.

§ 2. Preliminaries

We first introduce some function spaces. We denote by $C^\infty_{0,\sigma}(\mathbb{R}^3)$ the set of all C^∞ vector functions $v = (v_1, v_2, v_3)$ with compact support in $\mathbb{R}^3$ satisfying $\operatorname{div} v = 0$. $L^2_\sigma(\mathbb{R}^3)$ denotes the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3)$ with respect to the $L^2(\mathbb{R}^3)$-norm $\|\cdot\|_{L^2}$. Let $\mathscr{S}(\mathbb{R}^3)$ be the Schwartz class, and let $\mathscr{S}'(\mathbb{R}^3)$ be the space of tempered distributions. For $f \in \mathscr{S}(\mathbb{R}^3)$, the Fourier transform and the inverse Fourier transform are defined by

$$\mathscr{F}[f](\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^3,$$
$$\mathscr{F}^{-1}[f](x) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi} f(\xi)d\xi, \quad x \in \mathbb{R}^3,$$

respectively. Next, we recall the definition of the Littlewood–Paley decomposition. Let φ_0 be a function in $\mathscr{S}(\mathbb{R}^3)$ satisfying $0 \leqslant \widehat{\varphi_0}(\xi) \leqslant 1$ for all $\xi \in \mathbb{R}^3$, $\operatorname{supp} \widehat{\varphi_0} \subset \{\xi \in \mathbb{R}^3 \mid 1/2 \leqslant |\xi| \leqslant 2\}$ and

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi_j}(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $\varphi_j(x) := 2^{3j}\varphi_0(2^j x)$. We set $\widehat{\chi}(\xi) := 1 - \sum_{j\geqslant 1}\widehat{\varphi_j}(\xi)$. Let $\{\Delta_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley operator defined by $\Delta_j f := \varphi_j * f$ for $f \in \mathscr{S}'(\mathbb{R}^3)$. Then, we recall the definitions of the inhomogeneous and the homogeneous Besov spaces.

Definition 2.1. (i) For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^3)$ is defined to be the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{B^s_{p,q}} := \|\chi * f\|_{L^p} + \left\|\left\{2^{sj}\|\Delta_j f\|_{L^p}\right\}_{j=1}^{\infty}\right\|_{\ell^q} < \infty.$$

(ii) For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined to be the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{B}^s_{p,q}} := \left\|\left\{2^{sj}\|\Delta_j f\|_{L^p}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q} < \infty.$$

Let $H^s(\mathbb{R}^3)$ denote the Sobolev space of order $s \in \mathbb{R}$ with the inner product

$$\langle f, g\rangle_{H^s} := \int_{\mathbb{R}^3} (1-\Delta)^{\frac{s}{2}} f(x)\overline{(1-\Delta)^{\frac{s}{2}} g(x)}dx = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}(1+|\xi|^2)^s\widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$$

and the norm $\|f\|_{H^s} := \sqrt{\langle f, f\rangle_{H^s}}$. For $s > 0$, it is known that the norm equivalence

$$\|f\|_{H^s} \sim \|f\|_{L^2} + \|f\|_{\dot{H}^s}$$

holds, where $\|\cdot\|_{\dot{H}^s}$ denotes the homogeneous Sobolev semi-norm defined by

$$\|f\|_{\dot{H}^s} := \|f\|_{\dot{B}^s_{2,2}} = \left(\sum_{j\in\mathbb{Z}} 2^{2sj}\|\Delta_j f\|^2_{L^2}\right)^{\frac{1}{2}}.$$

We end this section by preparing some key inequalities for the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 2.2 ([17], [18]). *For $s > 3/2$, there exists a positive constant $C = C(s)$ such that*

$$\|f\|_{L^\infty} \leqslant C\left\{1 + \|f\|_{\dot{B}^0_{\infty,\infty}}\left(1 + \log^+\|f\|_{H^s}\right)\right\}$$

holds for all $f \in H^s(\mathbb{R}^3)$, where $\log^+ a := \max\{\log a, 0\}$ for $a > 0$.

Lemma 2.3 ([9], [6]). *For $s > 0$, there exists a positive constant $C = C(s)$ such that*

$$\|fg\|_{\dot{H}^s} \leqslant C\left(\|f\|_{L^\infty}\|g\|_{\dot{H}^s} + \|g\|_{L^\infty}\|f\|_{\dot{H}^s}\right)$$

holds for all $f, g \in L^\infty(\mathbb{R}^3) \cap \dot{H}^s(\mathbb{R}^3)$.

§ 3. Linear estimates

In this section, we consider the linear problem for the vorticity equations in the rotational framework. The linear equations for (E'_Ω) are described as

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Omega \mathbb{P}\left(e_3 \times u\right) = 0, \quad \operatorname{div} u = 0, \\ u(0,x) = \phi(x). \end{cases} \tag{3.1}$$

Taking curl to (3.1) and using the divergence-free condition, we have

$$\frac{\partial \omega}{\partial t} - \Omega \frac{\partial u}{\partial x_3} = 0, \quad \omega(0,x) = \psi(x), \tag{3.2}$$

where $\omega := \operatorname{curl} u = \nabla \times u$ and $\psi := \operatorname{curl} \phi$. By the Biot-Savart law, the gradient of the velocity ∇u has the representation in terms of the vorticity ω such that

$$\frac{\partial u}{\partial x_j} = \frac{\partial}{\partial x_j}(-\Delta)^{-1} \operatorname{curl} \omega = R_j \left(R \times \omega\right), \quad j = 1,2,3, \tag{3.3}$$

where $R = (R_1, R_2, R_3)$ and R_j denotes the Riesz transform in $\mathbb{R}^3$. Then the linear vorticity equations (3.2) can be rewritten as

$$\frac{\partial \omega}{\partial t} - \Omega \frac{\partial}{\partial x_3}(-\Delta)^{-1} \operatorname{curl} \omega = 0, \quad \omega(0,x) = \psi(x). \tag{3.4}$$

Taking the Fourier transform to (3.4) yields

$$\frac{\partial}{\partial t}\widehat{\omega}(t,\xi) - \Omega \frac{\xi_3}{|\xi|} \begin{pmatrix} 0 & \frac{\xi_3}{|\xi|} & -\frac{\xi_2}{|\xi|} \\ -\frac{\xi_3}{|\xi|} & 0 & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & -\frac{\xi_1}{|\xi|} & 0 \end{pmatrix} \widehat{\omega}(t,\xi) = 0, \quad \widehat{\omega}(0,\xi) = \widehat{\psi}(\xi). \tag{3.5}$$

Let us define

$$R(\xi) := \begin{pmatrix} 0 & \frac{\xi_3}{|\xi|} & -\frac{\xi_2}{|\xi|} \\ -\frac{\xi_3}{|\xi|} & 0 & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & -\frac{\xi_1}{|\xi|} & 0 \end{pmatrix}, \quad S(\xi) := \frac{\xi_3}{|\xi|} R(\xi)$$

for $\xi \in \mathbb{R}^3 \setminus \{0\}$. Then the solution to (3.5) is written as

$$\widehat{\omega}(t,\xi) = e^{\Omega t S(\xi)} \widehat{\psi}(\xi),$$

where $e^{\Omega t S(\xi)}$ is defined by the convergent series

$$e^{\Omega t S(\xi)} := \sum_{j=0}^{\infty} \frac{1}{j!} \left(\Omega t\right)^j S(\xi)^j \quad \text{on} \ \{\xi\}^{\perp}.$$

Let I be the 3×3 identity matrix. Note that since it holds

$$S(\xi)^2 v(\xi) = -\frac{\xi_3^2}{|\xi|^2} I v(\xi)$$

for $v(\xi) \in \mathbb{R}^3$ with $\xi \cdot v(\xi) = 0$, the solution of (3.5) is explicitly given by

$$\begin{aligned} \widehat{\omega}(t,\xi) &= \cos\left(\Omega t \frac{\xi_3}{|\xi|}\right) I \widehat{\psi}(\xi) + \sin\left(\Omega t \frac{\xi_3}{|\xi|}\right) R(\xi)\widehat{\psi}(\xi) \\ &= \frac{1}{2} e^{i\Omega t \frac{\xi_3}{|\xi|}} \{I - iR(\xi)\}\widehat{\psi}(\xi) + \frac{1}{2} e^{-i\Omega t \frac{\xi_3}{|\xi|}} \{I + iR(\xi)\}\widehat{\psi}(\xi). \end{aligned} \tag{3.6}$$

We remark that the explicit formula (3.6) has already been derived in [1] [12] [13] for the original equations of velocity fields. By (3.6), we have the following proposition.

Proposition 3.1. *For every $\Omega \in \mathbb{R}$ and every $\psi \in L^2_\sigma(\mathbb{R}^3)$, there exists a unique solution ω to* (3.4) *which is given explicitly by*

$$\omega(t,x) = T(\Omega t)\psi(x) := \frac{1}{2} e^{i\Omega t \frac{D_3}{|D|}} (I + \mathcal{R})\psi(x) + \frac{1}{2} e^{-i\Omega t \frac{D_3}{|D|}} (I - \mathcal{R})\psi(x) \tag{3.7}$$

for $t \geqslant 0$ and $x \in \mathbb{R}^3$, where

$$e^{\pm it\frac{D_3}{|D|}} f(x) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}} \widehat{f}(\xi) d\xi, \quad \mathcal{R} := \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix}.$$

We end this section by recalling the linear estimates for $T(t)$ given in (3.7). Since the phase $\xi_3/|\xi|$ is homogeneous function of degree 0, by the Littlewood-Paley decomposition and scaling, the matter is reduced to the frequency localized case. Now let us consider the operator

$$\mathscr{G}_\pm(t) f(x) := \int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it\frac{\xi_3}{|\xi|}} \widehat{\Phi}(\xi)\widehat{f}(\xi) d\xi, \quad (t,x) \in \mathbb{R}^{1+3},$$

where $\Phi \in \mathscr{S}(\mathbb{R}^3)$ satisfies $\operatorname{supp}\widehat{\Phi} \subset \{2^{-2} \leqslant |\xi| \leqslant 2^2\}$ and $\widehat{\Phi} = 1$ on $\{2^{-1} \leqslant |\xi| \leqslant 2\}$. The sharp Strichartz estimates for $\mathscr{G}_\pm(t)$ were obtained in [16]:

Theorem 3.2 ([16]). *Let $2 \leqslant q, r \leqslant \infty$ with $(q,r) \neq (2,\infty)$. Then the space-time estimates*

$$\|\mathscr{G}_\pm(t) f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2}$$

holds if and only if

$$\frac{1}{q} + \frac{1}{r} \leqslant \frac{1}{2}.$$

Corollary 3.3. *Let $2 \leqslant q, r \leqslant \infty$ satisfy $(q,r) \neq (2,\infty)$ and $\dfrac{1}{q} + \dfrac{1}{r} \leqslant \dfrac{1}{2}$. Then, there exists a positive constant $C = C(q,r)$ such that*

$$\left\| \Delta_j e^{\pm i\Omega t \frac{D_3}{|D|}} f \right\|_{L^q_t L^r_x} \leqslant C|\Omega|^{-\frac{1}{q}} \left(2^j\right)^{\frac{3}{2}-\frac{3}{r}} \|\Delta_j f\|_{L^2}$$

holds for all $\Omega \in \mathbb{R} \setminus \{0\}, j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R}^3)$.

For the proof, see [20].

§ 4. Local existence and blow-up criteria

In this section, let us recall the local theory for (E'_Ω). In particular, we shall review the unique existence results of local in time solutions and the blow-up criteria of the Beale-Kato-Majda type [4]. We remark that the local theory for (E'_Ω) in H^s-framework can be proved by the almost same strategy for the original Euler equations (E'_0) since the Coriolis force $\Omega\mathbb{P}(e_3 \times u)$ does not affect the energy due to its linearity and skew-symmetry:

$$\int_{\mathbb{R}^3} \Omega e_3 \times u(t,x) \cdot u(t,x) dx = 0.$$

We first recall the uniform local existence results with respect to the Coriolis parameter $\Omega \in \mathbb{R}$.

Theorem 4.1. *Let $s \in \mathbb{R}$ satisfy $s > 5/2$. Then, for every $\phi \in H^s(\mathbb{R}^3)$ satisfying $\operatorname{div}\phi = 0$, there exists a positive time $T_0 = T_0(\|\phi\|_{H^s})$ such that (E'_Ω) admits a unique classical solution u for all $\Omega \in \mathbb{R}$ in the class*

$$u \in C([0,T_0]; H^s(\mathbb{R}^3)) \cap C^1([0,T_0]; H^{s-1}(\mathbb{R}^3)).$$

For the proof, we refer to [14] and [16]. Next, let us recall the blow-up criteria for the local solutions to (E'_Ω).

Lemma 4.2. *Let $s > 5/2, \Omega \in \mathbb{R}$ and $\phi \in H^s(\mathbb{R}^3)$ with $\operatorname{div}\phi = 0$. Let u be the solution to (E'_Ω) in the class $C([0,T); H^s(\mathbb{R}^3)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^3))$ with some $T > 0$. Then, there exists a positive constant $C = C(s)$ depending only on s such that*

$$\|u(t)\|_{H^s} \leqslant \|\phi\|_{H^s} \exp\left\{ C \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \right\} \tag{4.1}$$

and

$$\|u(t)\|_{H^s} + e \leqslant (\|\phi\|_{H^s} + e)^{\alpha(t)} \exp\{Ct\alpha(t)\} \tag{4.2}$$

holds for all $0 \leqslant t < T$, where

$$\alpha(t) := \exp\left\{ C \int_0^t \|\operatorname{curl} u(\tau)\|_{\dot{B}^0_{\infty,\infty}} d\tau \right\}.$$

The first a priori estimate (4.1) is a standard one for the transport equations. The improved estimate (4.2) in terms of the vorticity was established by Beale-Kato-Majda [4] and further extentions were given by Kozono-Taniuchi [17] and Kozono-Ogawa-Taniuchi [18] by using the logarithmic Sobolev inequality (Lemma 2.2). By the standard argument of continuation of local solutions, Lemma 4.2 yields the following blow-up criteria.

Lemma 4.3. *Let $s > 5/2, \Omega \in \mathbb{R}$ and $\phi \in H^s(\mathbb{R}^3)$ with $\operatorname{div}\phi = 0$. Let u be the solution to (E'_Ω) in the class $C([0,T); H^s(\mathbb{R}^3)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^3))$. Assume that T is maximal, that is, u cannot be continued to the solution in the class $C([0,T'); H^s(\mathbb{R}^3)) \cap C^1([0,T'); H^{s-1}(\mathbb{R}^3))$ for any $T' > T$. Then, it holds*

$$\int_0^T \|\nabla u(t)\|_{L^\infty}\, dt = \int_0^T \|\operatorname{curl} u(t)\|_{\dot{B}^0_{\infty,\infty}}\, dt = \infty.$$

For the proofs of Lemmas 4.2 and 4.3, we refer to [4], [17], [18], [16] and [20].

§ 5. Proof of Theorem 1.1

We shall prove that the local solution u to (E'_Ω) constructed in Theorem 4.1 with $s > 7/2$ can be extended to any time interval $[0,T]$ provided the speed of rotation is sufficiently high. To this end, we adapt the argument in [7] [11] [16].

Let $s > 5/2$, and let $\phi \in H^{s+1}(\mathbb{R}^3)$ satisfy $\operatorname{div}\phi = 0$. Suppose that u be the solution to (E'_Ω) in the class $C([0,T_\Omega); H^{s+1}(\mathbb{R}^3)) \cap C^1([0,T_\Omega); H^s(\mathbb{R}^3))$, where $0 < T_\Omega < \infty$ denotes the maximal time of existence. Let us define

$$U(t) := \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau, \quad 0 \leqslant t < T_\Omega.$$

Then, it follows from the argument in Section 8 of [16] that for every $2 < q < \infty$ there exist positive constants $C_1 = C_1(s,q)$ and $C_2 = C_2(s)$ such that

$$U(t) \leqslant C_1 t^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \|\phi\|_{H^{s+1}} \left(1 + \|\phi\|_{H^{s+1}} t e^{C_2 U(t)}\right) \tag{5.1}$$

for all $0 \leqslant t < T_\Omega$. Now, for given time $0 < T < \infty$, we define

$$X_{T,\Omega} := \left\{ t \in [0,T] \cap [0,T_\Omega) \;\middle|\; U(t) \leqslant 1 \right\}, \quad T^*_\Omega := \sup X_{T,\Omega}.$$

We shall prove that $T^*_\Omega = \min\{T, T_\Omega\}$ when $|\Omega|$ is sufficiently large by contradiction argument. Assume that $T^*_\Omega < \min\{T, T_\Omega\}$. Then, we can take $\widetilde{T}$ satisfying $T^*_\Omega < \widetilde{T} < \min\{T, T_\Omega\}$. Since u belongs to $C([0,\widetilde{T}]; H^{s+1}(\mathbb{R}^3))$, we see that $U(t)$ is uniformly continuous on $[0,\widetilde{T}]$, and then it holds

$$U(T^*_\Omega) \leqslant 1. \tag{5.2}$$

Since $T_\Omega^* < \min\{T, T_\Omega\} \leqslant T$, it follows from (5.1) and (5.2) that

$$\begin{aligned} U(T_\Omega^*) &\leqslant C_1 (T_\Omega^*)^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \|\phi\|_{H^{s+1}} \left(1 + \|\phi\|_{H^{s+1}} T_\Omega^* e^{C_2 U(T_\Omega^*)}\right) \\ &\leqslant C_1 T^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \|\phi\|_{H^{s+1}} \left(1 + \|\phi\|_{H^{s+1}} T e^{C_2}\right). \end{aligned} \tag{5.3}$$

Hence taking a sufficiently large $\Omega \in \mathbb{R} \setminus \{0\}$ so that

$$|\Omega| \geqslant (2C_1)^q T^{q-1} \left\{ \|\phi\|_{H^{s+1}} \left(1 + e^{C_2} T \|\phi\|_{H^{s+1}}\right) \right\}^q, \tag{5.4}$$

we have by (5.3) that

$$U(T_\Omega^*) \leqslant \frac{1}{2}.$$

Then, one can take a time S such that $T_\Omega^* < S < \widetilde{T}$ and $U(S) \leqslant 1$, which contradicts the definition of T_Ω^*. Therefore, we have $T_\Omega^* = \min\{T, T_\Omega\}$ provided the speed of rotation $\Omega \in \mathbb{R} \setminus \{0\}$ satisfies (5.4).

Let $\Omega \in \mathbb{R} \setminus \{0\}$ satisfy (5.4), and assume that $T_\Omega < T$. Then it follows from the above argument that $T_\Omega = T_\Omega^* = \sup X_{T,\Omega}$. Therefore we have

$$U(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \leqslant 1 < \infty$$

for all $0 \leqslant t < T_\Omega$. However, by Lemma 4.3, this contradicts the maximality of T_Ω. Hence we obtain $T_\Omega \geqslant T$ if the speed of rotation $\Omega \in \mathbb{R} \setminus \{0\}$ is high enough as in (5.4). This completes the proof of Theorem 1.1.

§ 6. Proof of Theorem 1.2

The strategy of the proof is same as that of Theorem 1.1. In order to treat the case $s = 7/2$, we use the critical Sobolev embeddings $\dot{B}^0_{\infty,2}(\mathbb{R}^3) \hookrightarrow \mathrm{BMO}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$ instead of $B^\varepsilon_{\infty,\infty}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ for $\varepsilon > 0$, and the refined blow-up criterion (4.2) in terms of the vorticity.

Let $\phi \in H^{\frac{7}{2}}(\mathbb{R}^3)$ with $\operatorname{div} \phi = 0$, and let u be the solution to (E'_Ω) in the class $u \in C([0, T_\Omega); H^{\frac{7}{2}}(\mathbb{R}^3)) \cap C^1([0, T_\Omega); H^{\frac{5}{2}}(\mathbb{R}^3))$, where $0 < T_\Omega < \infty$ denotes the maximal time of existence. Taking curl to (E'_Ω) and using the Biot-Savart law (3.3), we have the vorticity equation

$$\begin{cases} \dfrac{\partial \omega}{\partial t} - \Omega \dfrac{\partial}{\partial x_3} (-\Delta)^{-1} \operatorname{curl} \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = 0, \\ \omega(0, x) = \psi(x), \end{cases} \tag{6.1}$$

where $\omega := \operatorname{curl} u = \nabla \times u$ and $\psi := \operatorname{curl} \phi$. By the Plancherel theorem and the Lebesgue dominated convergence theorem, we see that $\Omega \dfrac{\partial}{\partial x_3}(-\Delta)^{-1}$ curl is the infinitesimal generator of the C_0 semigroup $T(\Omega t)$ (defined in Proposition 3.1) on $L^2_\sigma(\mathbb{R}^3)$ with the domain of generator $L^2_\sigma(\mathbb{R}^3)$. Therefore by the Duhamel principle, the solution ω to (6.1) can be represented as

$$\begin{aligned}\omega(t) = T\,(\Omega t)\,\psi - \int_0^t T\,(\Omega(t-\tau))\,(u(\tau)\cdot\nabla)\,\omega(\tau)d\tau \\ + \int_0^t T\,(\Omega(t-\tau))\,(\omega(\tau)\cdot\nabla)\,u(\tau)d\tau\end{aligned} \tag{6.2}$$

for $0 < t < T_\Omega$. We shall derive the $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$-estimates for the vorticity ω. Let $2 < q < \infty$. Then, by the Minkowski inequality and Corollary 3.3, we have

$$\begin{aligned}\|T\,(\Omega t)\,\psi\|_{L^q_t(0,\infty;\dot{B}^0_{\infty,\infty})} &\leqslant \left\| \left(\sum_{j\in\mathbb{Z}} \|\Delta_j T\,(\Omega t)\,\psi\|^2_{L^\infty} \right)^{\frac{1}{2}} \right\|_{L^q_t(0,\infty)} \\ &\leqslant \left(\sum_{j\in\mathbb{Z}} \|\Delta_j T\,(\Omega t)\,\psi\|^2_{L^q_t(0,\infty;L^\infty)} \right)^{\frac{1}{2}} \\ &\leqslant C|\Omega|^{-\frac{1}{q}} \|\psi\|_{\dot{H}^{\frac{3}{2}}}\end{aligned} \tag{6.3}$$

with some constant $C = C(q) > 0$. Next, let us consider the Duhamel terms in (6.2). It follows from the Minkowski inequality and Corollary 3.3 that

$$\begin{aligned}&\left\| \Delta_j \int_0^t T\,(\Omega(t-\tau))\,(u(\tau)\cdot\nabla)\,\omega(\tau)d\tau \right\|_{L^q_t(0,T;L^\infty)} \\ &\leqslant \int_0^T \|\Delta_j T\,(\Omega(t-\tau))\,(u(\tau)\cdot\nabla)\,\omega(\tau)\|_{L^q_t(\tau,T;L^\infty)}\, d\tau \\ &\leqslant C|\Omega|^{-\frac{1}{q}} \int_0^T 2^{\frac{3}{2}j}\, \|\Delta_j\,(u(\tau)\cdot\nabla)\,\omega(\tau)\|_{L^2}\, d\tau\end{aligned} \tag{6.4}$$

for all $j \in \mathbb{Z}$ and $0 < T < T_\Omega$ with some constant $C = C(q) > 0$. Hence, by the Minkowski inequality and (6.4), we have

$$\begin{aligned}&\left\| \int_0^t T\,(\Omega(t-\tau))\,(u(\tau)\cdot\nabla)\,\omega(\tau)d\tau \right\|_{L^q_t(0,T;\dot{B}^0_{\infty,\infty})} \\ &\leqslant \left\| \left(\sum_{j\in\mathbb{Z}} \left\| \Delta_j \int_0^t T\,(\Omega(t-\tau))\,(u(\tau)\cdot\nabla)\,\omega(\tau)d\tau \right\|^2_{L^\infty} \right)^{\frac{1}{2}} \right\|_{L^q_t(0,T)} \\ &\leqslant \left(\sum_{j\in\mathbb{Z}} \left\| \Delta_j \int_0^t T\,(\Omega(t-\tau))\,(u(\tau)\cdot\nabla)\,\omega(\tau)d\tau \right\|^2_{L^q_t(0,T;L^\infty)} \right)^{\frac{1}{2}} \\ &\leqslant C|\Omega|^{-\frac{1}{q}} \left\{ \sum_{j\in\mathbb{Z}} \left(\int_0^T 2^{\frac{3}{2}j}\, \|\Delta_j\,(u(\tau)\cdot\nabla)\,\omega(\tau)\|_{L^2}\, d\tau \right)^2 \right\}^{\frac{1}{2}} \\ &\leqslant C|\Omega|^{-\frac{1}{q}} \int_0^T \|(u(\tau)\cdot\nabla)\,\omega(\tau)\|_{\dot{H}^{\frac{3}{2}}}\, d\tau.\end{aligned} \tag{6.5}$$

Similarly to (6.4) and (6.5), we also have

$$
\begin{aligned}
&\left\| \int_0^t T\left(\Omega(t-\tau)\right)\left(\omega(\tau)\cdot\nabla\right)u(\tau)d\tau \right\|_{L_t^q(0,T;\dot{B}^0_{\infty,\infty})} \\
&\qquad \leqslant C|\Omega|^{-\frac{1}{q}} \int_0^T \left\|\left(\omega(\tau)\cdot\nabla\right)u(\tau)\right\|_{\dot{H}^{\frac{3}{2}}} d\tau.
\end{aligned}
\tag{6.6}
$$

Therefore, by (6.2), (6.3), (6.5) and (6.6), we see that for every $2 < q < \infty$ there exists a positive constant $C = C(q)$ such that

$$
\begin{aligned}
&\|\omega\|_{L^q(0,T;\dot{B}^0_{\infty,\infty})} \\
&\leqslant C|\Omega|^{-\frac{1}{q}} \left\{ \|\psi\|_{\dot{H}^{\frac{3}{2}}} + \int_0^T \big(\left\|\left(u(\tau)\cdot\nabla\right)\omega(\tau)\right\|_{\dot{H}^{\frac{3}{2}}} + \left\|\left(\omega(\tau)\cdot\nabla\right)u(\tau)\right\|_{\dot{H}^{\frac{3}{2}}} \big) d\tau \right\}
\end{aligned}
\tag{6.7}
$$

for all $0 < T < T_\Omega$.

Now, let us define

$$
V(t) := \int_0^t \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}} d\tau, \quad 0 \leqslant t < T_\Omega.
$$

Since it holds

$$
\begin{aligned}
\left\|\left(u\cdot\nabla\right)\omega\right\|_{\dot{H}^{\frac{3}{2}}} + \left\|\left(\omega\cdot\nabla\right)u\right\|_{\dot{H}^{\frac{3}{2}}} &\leqslant C\big(\|\omega\otimes u\|_{\dot{H}^{\frac{5}{2}}} + \|u\otimes\omega\|_{\dot{H}^{\frac{5}{2}}} \big) \\
&\leqslant C\|\omega\|_{H^{\frac{5}{2}}} \|u\|_{H^{\frac{5}{2}}}
\end{aligned}
\tag{6.8}
$$

by the divergence-free conditions and Lemma 2.3, it follows from the Hölder inequality, (6.7), (6.8) and (4.2) that

$$
\begin{aligned}
V(t) &\leqslant t^{1-\frac{1}{q}} \|\omega\|_{L^q(0,t;\dot{B}^0_{\infty,\infty})} \\
&\leqslant Ct^{1-\frac{1}{q}}|\Omega|^{-\frac{1}{q}} \left(\|\psi\|_{\dot{H}^{\frac{3}{2}}} + \int_0^t \|\omega(\tau)\|_{H^{\frac{5}{2}}} \|u(\tau)\|_{H^{\frac{5}{2}}} d\tau \right) \\
&\leqslant Ct^{1-\frac{1}{q}}|\Omega|^{-\frac{1}{q}} \left(\|\phi\|_{H^{\frac{5}{2}}} + \int_0^t \|u(\tau)\|^2_{H^{\frac{7}{2}}} d\tau \right) \\
&\leqslant Ct^{1-\frac{1}{q}}|\Omega|^{-\frac{1}{q}} \left\{ \|\phi\|_{H^{\frac{5}{2}}} + \int_0^t \left(\|\phi\|_{H^{\frac{7}{2}}} + e\right)^{2\alpha(\tau)} \exp\left\{C\tau\alpha(\tau)\right\} d\tau \right\}.
\end{aligned}
$$

Hence, we see that there exist an absolute constant $C_* = C_*(7/2) > 0$ and a constant $C_q > 0$ depending on q such that

$$
V(t) \leqslant C_q t^{1-\frac{1}{q}}|\Omega|^{-\frac{1}{q}} \left\{ \|\phi\|_{H^{\frac{5}{2}}} + t\big(\|\phi\|_{H^{\frac{7}{2}}} + e\big)^{2\alpha(t)} \exp\left\{C_* t\alpha(t)\right\} \right\}
\tag{6.9}
$$

for all $0 \leqslant t < T_\Omega$, where

$$
\alpha(t) := \exp\left\{C_* V(t)\right\}.
$$

Now, for given time $0 < T < \infty$, we define

$$X_{T,\Omega} := \left\{ t \in [0, T] \cap [0, T_\Omega) \,\middle|\, V(t) \leqslant 1 \right\}, \quad T_\Omega^* := \sup X_{T,\Omega}.$$

We shall prove that $T_\Omega^* = \min\{T, T_\Omega\}$ when $|\Omega|$ is sufficiently large by contradiction argument. Assume that $T_\Omega^* < \min\{T, T_\Omega\}$. Then, we can take $\widetilde{T}$ satisfying $T_\Omega^* < \widetilde{T} < \min\{T, T_\Omega\}$. Since u belongs to $C([0, \widetilde{T}]; H^{\frac{7}{2}}(\mathbb{R}^3))$, we see that $V(t)$ is uniformly continuous on $[0, \widetilde{T}]$, and then it holds

$$V(T_\Omega^*) \leqslant 1. \tag{6.10}$$

Since $T_\Omega^* < \min\{T, T_\Omega\} \leqslant T$, it follows from (6.9) and (6.10) that

$$\begin{aligned} V(T_\Omega^*) &\leqslant C_q (T_\Omega^*)^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \left[\|\phi\|_{H^{\frac{5}{2}}} + T_\Omega^* \big(\|\phi\|_{H^{\frac{7}{2}}} + e \big)^{2\alpha(T_\Omega^*)} \exp\left\{ C_* T_\Omega^* \alpha(T_\Omega^*) \right\} \right] \\ &\leqslant C_q T^{1-\frac{1}{q}} |\Omega|^{-\frac{1}{q}} \left[\|\phi\|_{H^{\frac{5}{2}}} + T \big(\|\phi\|_{H^{\frac{7}{2}}} + e \big)^{2\exp\{C_*\}} \exp\left\{ C_* e^{C_* T} \right\} \right] \\ &\leqslant |\Omega|^{-\frac{1}{q}} C_q T^{1-\frac{1}{q}} \left\{ \|\phi\|_{H^{\frac{5}{2}}} + T \big(\|\phi\|_{H^{\frac{7}{2}}} + e \big)^{C_*'} e^{C_*' T} \right\}. \end{aligned} \tag{6.11}$$

Hence taking a sufficiently large $\Omega \in \mathbb{R} \setminus \{0\}$ so that

$$|\Omega| \geqslant (2C_q)^q T^{q-1} \left\{ \|\phi\|_{H^{\frac{5}{2}}} + T \big(\|\phi\|_{H^{\frac{7}{2}}} + e \big)^{C_*'} e^{C_*' T} \right\}^q, \tag{6.12}$$

by (6.11) we have

$$V(T_\Omega^*) \leqslant \frac{1}{2}.$$

Then, one can take a time S such that $T_\Omega^* < S < \widetilde{T}$ and $V(S) \leqslant 1$, which contradicts the definition of T_Ω^*. Therefore, we have $T_\Omega^* = \min\{T, T_\Omega\}$ provided the speed of rotation $\Omega \in \mathbb{R} \setminus \{0\}$ satisfies (6.12).

Let $\Omega \in \mathbb{R} \setminus \{0\}$ satisfy (6.12), and assume that $T_\Omega < T$. Then it follows from the above argument that $T_\Omega = T_\Omega^* = \sup X_{T,\Omega}$. Therefore we have

$$V(t) = \int_0^t \|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}} \, d\tau \leqslant 1 < \infty$$

for all $0 \leqslant t < T_\Omega$. However, by Lemma 4.3, this contradicts the maximality of T_Ω. Hence we obtain $T_\Omega \geqslant T$ if the speed of rotation $\Omega \in \mathbb{R} \setminus \{0\}$ is high enough as in (6.12). This completes the proof of Theorem 1.2.

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